

SPACELIKE HYPERSURFACES WITH NEGATIVE TOTAL ENERGY IN DE SITTER SPACETIME

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ABSTRACT. De Sitter spacetime can be separated into two parts along two kinds of hypersurfaces and the half-de Sitter spacetimes are covered by the planar and hyperbolic coordinates respectively. Two positive energy theorems were proved previously for certain \mathcal{P} -asymptotically de Sitter and \mathcal{H} -asymptotically de Sitter initial data sets by the second author and collaborators. These initial data sets are asymptotic to time slices of the two kinds of half-de Sitter spacetimes respectively, and their mean curvatures are bounded from above by certain constants. While the mean curvatures violate these conditions, the spacelike hypersurfaces with negative total energy in the two kinds of half-de Sitter spacetimes are constructed in this short paper.

1. INTRODUCTION

In general relativity, the positive energy theorem plays a fundamental role which serves as a consistent verification of the theory. If the positive energy theorem holds, the spacetime with vanishing total energy can be viewed as the ground state. In the case of zero cosmological constant, the positive energy theorem for asymptotically flat spacetimes was firstly proved by Schoen and Yau [9, 10, 11], and then by Witten [13] using a different method. Recently, Witten's method was extended successfully to asymptotically Anti-de Sitter spacetimes and the positive energy theorem was proved completely and rigorously in the case of negative cosmological constant [12, 5, 16, 8, 15].

Recent cosmological observations indicated that our universe has a positive cosmological constant. It is therefore important to study whether the positive energy theorem for asymptotically de Sitter spacetimes holds. De Sitter spacetime with cosmological constant $\Lambda = \frac{3}{\lambda^2} > 0$, ($\lambda > 0$) is a hypersurface embedded into 5-dimensional Minkowski spacetime $\mathbb{R}^{1,4}$

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \frac{3}{\Lambda} \quad (1.1)$$

with the induced metric (cf. [6]). It is covered by global coordinates where each time slice is a 3-sphere with constant curvature and has no spatial infinity. As the corresponding Killing vector fields to the Lorentzian generators are timelike in some region of de Sitter spacetime and spacelike in some other region, there is no positive conserved energy in de Sitter spacetime [14, 1]. So the issue of the positive energy theorem for asymptotically de Sitter spacetimes becomes sophisticated.

However, de Sitter spacetime can be separated into two parts along the hypersurface $X^0 = X^4$ and the half-de Sitter spacetime is covered by the planar coordinates with the de Sitter metric

$$\tilde{g}_{dS}^{\mathcal{P}} = -dt^2 + e^{\frac{2t}{\lambda}} \left((dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right). \quad (1.2)$$

Similarly, de Sitter spacetime can be separated into two parts along the hypersurface $X^4 = -\lambda$ and the half-de Sitter spacetime is covered by the hyperbolic coordinates with the de Sitter metric

$$\tilde{g}_{dS}^{\mathcal{H}} = -dT^2 + \sinh^2 \frac{T}{\lambda} \left(dR^2 + \lambda^2 \sinh^2 \frac{R}{\lambda} (d\theta^2 + \sin^2 \theta d\psi^2) \right). \quad (1.3)$$

In [7], two positive energy theorems were proved for certain \mathcal{P} -asymptotically de Sitter and \mathcal{H} -asymptotically de Sitter initial data sets. These initial data sets are asymptotic to time slices of the two kinds of half-de Sitter spacetimes covered by the planar coordinates and hyperbolic coordinates respectively. And their mean curvatures satisfy (2.1) for \mathcal{P} -asymptotically de Sitter initial data sets and satisfy (2.2) for \mathcal{H} -asymptotically de Sitter initial data sets.

In the case of zero or negative cosmological constants, the positive energy theorem holds for any asymptotically flat or asymptotically anti-de Sitter spacelike hypersurfaces whose mean curvatures have no restriction as (2.1) or (2.2). This motivates us to study further the special feature of the conditions (2.1) and (2.2) in the case of positive cosmological constant. In this short paper, we shall construct spacelike hypersurfaces with negative total energy in the two kinds of half-de Sitter spacetimes. The constructed spacelike hypersurfaces are either \mathcal{P} -asymptotically de Sitter or \mathcal{H} -asymptotically de Sitter. However, their mean curvatures violate (2.1) in the \mathcal{P} -asymptotically de Sitter case and violate (2.2) in the \mathcal{H} -asymptotically de Sitter case. As the two kinds of half-de Sitter spacetimes satisfy the vacuum Einstein field equations with positive cosmological constant, the dominant energy condition holds automatically. So these examples provide counterexamples of the positive energy theorem for general asymptotically de Sitter spacelike hypersurfaces. Thus it indicates that the positive energy theorem is no longer the general feature in theory of gravity when the cosmological constant is positive.

We would like to point out that de Sitter spacetime can be also separated into two parts along the hypersurface $X^3 = 0$. The time slices in this half-de Sitter spacetime are hemispheres. The counterexample to Min-Oo's conjecture related to the rigidity of hemispheres, constructed by Brendle, Marques and Neves, can be understood as the another failure of the positive energy theorem for spacetimes with positive cosmological constant [3]. We refer to the survey paper [4] for the recent works on Min-Oo's conjecture.

2. POSITIVE ENERGY THEOREMS

In this section, we review two positive energy theorems proved in [7]. Let $(N^{1,3}, \tilde{g})$ be a spacetime satisfying the Einstein equations with a positive cosmological constant $\Lambda = \frac{3}{\lambda^2}$. Let (M, g, K) be a spacelike hypersurface with induced Riemannian metric g and second fundamental form K . It is \mathcal{P} -asymptotically de

Sitter of order $\tau > \frac{1}{2}$ if there is a compact set M_c such that $M - M_c$ is the disjoint union of a finite number of subsets M_1, \dots, M_k - called the “ends” of M - each diffeomorphic to $\mathbb{R}^3 - B_r$ where B_r is the closed ball of radius r with center at the coordinate origin. And g and $h = K - \sqrt{\frac{\Lambda}{3}}g$ satisfy

$$g = \mathcal{P}^2 \bar{g}, \quad h = \mathcal{P} \bar{h}, \quad \bar{g} - \check{g} \in C_{-\tau}^{2,\alpha}(M), \quad \bar{h} \in C_{-\tau-1}^{1,\alpha}(M)$$

for certain $\tau > \frac{1}{2}$, $0 < \alpha < 1$, where \mathcal{P} is certain positive constant, \check{g} is the standard metric of \mathbb{R}^3 , $C_{-\tau}^{k,\alpha}$ are weighted Hölder spaces defined in [2]. Moreover, the scalar curvature $\mathcal{R}_{\bar{g}} \in L^1(M)$, $T_{0i} \in L^1(M)$. Let the metric $\check{g}_{\mathcal{P}} = \mathcal{P}^2 \check{g}$. Let $\{x^i\}$ be natural coordinates of \mathbb{R}^3 , $\bar{g}_{ij} = \bar{g}(\partial_i, \partial_j)$, $\bar{h}_{ij} = \bar{h}(\partial_i, \partial_j)$. The total energy E_l and the total linear momentum P_{lk} of M_l for \mathcal{P} -asymptotically de Sitter initial data set (M, g, K) are

$$E_l = \frac{\mathcal{P}}{16\pi} \lim_{r \rightarrow \infty} \int_{S_{r,l}} (\partial_j \bar{g}_{ij} - \partial_i \bar{g}_{jj}) * dx^i,$$

$$P_{lk} = \frac{\mathcal{P}^2}{8\pi} \lim_{r \rightarrow \infty} \int_{S_{r,l}} (\bar{h}_{ki} - \bar{g}_{ki} tr_{\bar{g}}(\bar{h})) * dx^i.$$

An initial data set (M, g, K) is \mathcal{H} -asymptotically de Sitter of order $\tau > \frac{3}{2}$ if there is a compact set M_c such that $M - M_c$ is the disjoint union of a finite number of subsets M_1, \dots, M_k - called the “ends” of M - each diffeomorphic to $\mathbb{R}^3 - B_r$ where B_r is the closed ball of radius r with center at the coordinate origin. And g and $h = K - \frac{\coth \frac{T}{\lambda}}{\lambda} g$ satisfy

$$g = \mathcal{H}^2 \bar{g}, \quad h = \mathcal{H} \bar{h}, \quad a_{ij} = O(e^{-\frac{T}{\lambda} r}), \quad \check{\nabla}_k^{\mathcal{H}} a_{ij} = O(e^{-\frac{T}{\lambda} r}),$$

$$\check{\nabla}_l^{\mathcal{H}} \check{\nabla}_k^{\mathcal{H}} a_{ij} = O(e^{-\frac{T}{\lambda} r}), \quad \bar{h}_{ij} = O(e^{-\frac{T}{\lambda} r}), \quad \check{\nabla}_k^{\mathcal{H}} \bar{h}_{ij} = O(e^{-\frac{T}{\lambda} r})$$

on each end for certain constant $T \neq 0$, where $\mathcal{H} = \sinh \frac{T}{\lambda}$, $a_{ij} = \bar{g}(\check{e}_i, \check{e}_j) - \check{g}_{\mathcal{H}}(\check{e}_i, \check{e}_j)$, $\bar{h}_{ij} = \bar{h}(\check{e}_i, \check{e}_j)$. And $\check{g}_{\mathcal{H}}$ is the hyperbolic metric

$$\check{g}_{\mathcal{H}} = dR^2 + \lambda^2 \sinh^2 \frac{R}{\lambda} (d\theta^2 + \sin^2 \theta d\psi^2).$$

$\check{e}_1 = \partial_R$, $\check{e}_2 = \frac{\partial_\theta}{\lambda \sinh \frac{R}{\lambda}}$, $\check{e}_3 = \frac{\partial_\psi}{\lambda \sinh \frac{R}{\lambda} \sin \theta}$ is the frame, $\{\check{e}^i\}$ is the coframe. $\check{\nabla}^{\mathcal{H}}$ is the Levi-Civita connection of $\check{g}_{\mathcal{H}}$. Let $\bar{\mathcal{R}}$, $\bar{\nabla}$, ρ_z be the scalar curvature, the Levi-Civita connection of \bar{g} and the distance function with respect to $z \in M$ respectively. We further require $(\bar{\mathcal{R}} + \frac{6}{\lambda^2}) e^{\frac{\rho_z}{\lambda}} \in L^1(M)$, $(\bar{\nabla}^j \bar{h}_{ij} - \bar{\nabla}_i tr_{\bar{g}}(\bar{h})) e^{\frac{\rho_z}{\lambda}} \in L^1(M)$. Denote

$$\mathcal{E}_l = \check{\nabla}^{\mathcal{H},j} \bar{g}_{1j} - \check{\nabla}_1^{\mathcal{H}} tr_{\check{g}_{\mathcal{H}}}(\bar{g}) + \frac{1}{\lambda} (a_{22} + a_{33}) + 2(\bar{h}_{22} + \bar{h}_{33}).$$

The total energy-momentum of the end M_l are

$$E_{lv}^{\mathcal{H}} = \frac{\mathcal{H}^2}{16\pi} \lim_{R \rightarrow \infty} \int_{S_{R,l}} \mathcal{E}_l n^\nu e^{\frac{R}{\lambda}} \check{e}^2 \wedge \check{e}^3$$

where $0 \leq \nu \leq 3$, $(n^\nu) = (1, \sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta)$, $S_{R,l}$ is the coordinate sphere of radius R in M_l .

The following two positive energy theorems are proved in [7].

Theorem 2.1. *Let (M, g, K) be an \mathcal{P} -asymptotically de Sitter initial data set of order $1 \geq \tau > \frac{1}{2}$ which has possibly a finite number of apparent horizons in spacetime $(N^{1,3}, \tilde{g})$ with positive cosmological constant $\Lambda > 0$. Suppose $N^{1,3}$ satisfies the dominant energy condition. If the trace of the second fundamental form of M satisfies*

$$tr_g(K) \leq \sqrt{3\Lambda}, \quad (2.1)$$

then for each end M_l ,

$$E_l \geq |P_l|_{\check{g}_\mathcal{P}}.$$

If $E_l = 0$ for some end M_{l_0} , then

$$(M, g, K) \equiv \left(\mathbb{R}^3, \mathcal{P}^2 \check{g}, \sqrt{\frac{\Lambda}{3}} \mathcal{P}^2 \check{g} \right).$$

Moreover, the mean curvature achieves the equality in (2.1) and the spacetime $(N^{1,3}, \tilde{g})$ is de Sitter along M . In particular, $(N^{1,3}, \tilde{g})$ is globally de Sitter in the planar coordinates if it is globally hyperbolic.

Theorem 2.2. *Let (M, g, K) be an \mathcal{H} -asymptotically de Sitter initial data set of order $\tau > \frac{3}{2}$ which has possibly a finite number of apparent horizons in spacetime $(N^{1,3}, \tilde{g})$ with positive cosmological constant $\Lambda > 0$. Suppose $N^{1,3}$ satisfies the dominant energy condition. If the trace of the second fundamental form of M satisfies*

$$tr_g(K) \sinh \frac{T}{\lambda} \leq \sqrt{3\Lambda} \cosh \frac{T}{\lambda}, \quad (2.2)$$

then for each end M_l ,

$$E_{l_0}^{\mathcal{H}} \geq \sqrt{(E_{l_1}^{\mathcal{H}})^2 + (E_{l_2}^{\mathcal{H}})^2 + (E_{l_3}^{\mathcal{H}})^2}.$$

If $E_{l_0}^{\mathcal{H}} = 0$ for some end M_{l_0} , then

$$(M, g, K) \equiv \left(\mathbb{H}^3, \sinh^2 \frac{T}{\lambda} \check{g}_{\mathcal{H}}, \sqrt{\frac{\Lambda}{3}} \sinh \frac{T}{\lambda} \cosh \frac{T}{\lambda} \check{g}_{\mathcal{H}} \right).$$

Moreover, the mean curvature achieves the equality in (2.2) and the spacetime $(N^{1,3}, \tilde{g})$ is de Sitter along M . In particular, $(N^{1,3}, \tilde{g})$ is globally de Sitter in the hyperbolic coordinates if it is globally hyperbolic.

In general, the conformal factors \mathcal{P} and \mathcal{H} are functions on 3-manifolds. The above positive energy theorems hold only when \mathcal{P} and \mathcal{H} are constants on ends. Let e_0 be timelike unit normal to M . It is well-known that

$$K = \frac{1}{2} (\mathcal{L}_{e_0} \tilde{g})^{proj}$$

where \mathcal{L} is the Lie derivative and “*proj*” means the projection to the tangent bundle of M . For t -slice with $e_0 = \partial_t$ in the metric (1.2),

$$tr_g(K) = \sqrt{3\Lambda},$$

and for T -slice ($T \neq 0$) with $e_0 = \partial_T$ in the metric (1.3),

$$tr_g(K) = \sqrt{3\Lambda} \coth \frac{T}{\lambda}.$$

Thus the mean curvatures conditions (2.1) and (2.2) indicate that, in certain sense of average, the 3-space evolution in spacetime N should not be too rapid to exceed the standard de Sitter spacetime in order to keep positivity of the total energy.

3. NEGATIVE TOTAL ENERGY

In this section, we construct certain spacelike hypersurfaces with negative total energy. The mean curvatures of these hypersurfaces violate the condition (2.1) or (2.2). These hypersurfaces are constructed by slightly perturbing the t or T -slices in half-de Sitter spacetime.

First of all, we provide the formula of the second fundamental form of a graph. Let (M^n, \hat{g}) be an n -dimensional Riemannian manifold, $\hat{\nabla}$ be its Levi-Civita connection, I be an interval of \mathbb{R} . We equip $I \times M$ with the wrapped product

$$\tilde{g} = -dt^2 + \rho^2 \hat{g},$$

where $\rho = \rho(t)$ is a positive smooth function. Suppose that Σ is a smooth graph which is given by

$$\begin{aligned} F : M &\longrightarrow I \times M \\ x &\longmapsto (f(x), x). \end{aligned}$$

Choosing a frame $\{\hat{e}_i\}$ on M , the tangent space of the graph Σ is spanned by

$$F_* \hat{e}_i = \hat{\nabla}_i f \partial_t + \hat{e}_i, \quad i = 1, \dots, n.$$

As $\partial_t + \rho^{-2} \hat{\nabla} f$ is orthogonal to Σ , the graph is spacelike if and only if

$$\rho^2 > |\hat{\nabla} f|_{\hat{g}}^2. \quad (3.1)$$

Then, $\nu = \frac{\partial_t + \rho^{-2} \hat{\nabla} f}{\sqrt{1 - \rho^{-2} |\hat{\nabla} f|_{\hat{g}}^2}}$ is a timelike unit normal vector field of the graph. Let g be the induced metric on Σ . The components of the second fundamental form with respect to the normal vector ν are

$$\begin{aligned} K_{ij} &= K(F_* \hat{e}_i, F_* \hat{e}_j) = \mu \left(\hat{\nabla}_{i,j}^2 f - 2\rho' \rho^{-1} \hat{\nabla}_i f \hat{\nabla}_j f + \rho' \rho \hat{g}_{ij} \right), \\ tr_g(K) &= \mu \rho^{-2} \left(\hat{g}^{ij} + \mu^2 \rho^{-2} \hat{\nabla}^i f \hat{\nabla}^j f \right) \hat{\nabla}_{i,j}^2 f - \mu^3 \rho' \rho^{-1} + (n+1) \mu \rho' \rho^{-1}, \end{aligned}$$

where $\mu = (1 - \rho^{-2} |\hat{\nabla} f|_{\hat{g}}^2)^{-\frac{1}{2}}$.

(i) *The \mathcal{P} -asymptotically de Sitter case:* Consider the graph where

$$f(x) = t_0 + \varepsilon (1 + r^2)^{-\frac{1}{2}}$$

for certain constant t_0 in the half-de Sitter spacetime equipped with the metric (1.2) in the planar coordinates. As

$$\rho^{-2}|\check{\nabla}f|_{\check{g}}^2 = \varepsilon^2 (1+r^2)^{-3} r^2 e^{-\frac{2}{\lambda}(t_0+\varepsilon(1+r^2)^{-\frac{1}{2}})},$$

(3.1) holds and the graph is an embedding spacelike hypersurface for ε is sufficiently small.

Note that

$$\begin{aligned}\partial_i f &= -\varepsilon (1+r^2)^{-\frac{3}{2}} x_i, \\ \partial_{i,j}^2 f &= 3\varepsilon (1+r^2)^{-\frac{5}{2}} x_i x_j - \varepsilon (1+r^2)^{-\frac{3}{2}} \delta_{ij},\end{aligned}$$

thus the induced metric and the second fundamental forms of the graph are

$$\begin{aligned}g_{ij} &= e^{\frac{2f}{\lambda}} \delta_{ij} - \varepsilon^2 (1+r^2)^{-3} x_i x_j, \\ K_{ij} &= \mu \left(\frac{1}{\lambda} e^{\frac{2f}{\lambda}} \delta_{ij} - \varepsilon (1+r^2)^{-\frac{3}{2}} \delta_{ij} - \frac{2}{\lambda} \varepsilon^2 (1+r^2)^{-3} x_i x_j + 3\varepsilon (1+r^2)^{-\frac{5}{2}} x_i x_j \right),\end{aligned}$$

where $\mu = \left(1 - \varepsilon^2 r^2 (1+r^2)^{-3} e^{-\frac{2f}{\lambda}} \right)^{-\frac{1}{2}} = 1 + \frac{1}{2} \varepsilon^2 r^2 (1+r^2)^{-3} e^{-\frac{2f}{\lambda}} + O(r^{-8})$. Set $\mathcal{P} = e^{\frac{t_0}{\lambda}}$, $\bar{g} = \mathcal{P}^{-2} g$, $a = \bar{g} - \check{g}$ and $\bar{h} = \mathcal{P}^{-1} \left(K - \frac{1}{\lambda} g \right)$. We have,

$$\begin{aligned}a_{ij} &= \left(e^{\frac{2(f-t_0)}{\lambda}} - 1 \right) \delta_{ij} - \mathcal{P}^{-2} \varepsilon^2 (1+r^2)^{-3} x_i x_j = \left(e^{\frac{2(f-t_0)}{\lambda}} - 1 \right) \delta_{ij} + O(r^{-4}), \\ \partial_k a_{ij} &= O(r^{-2}), \quad \partial_k \partial_l a_{ij} = O(r^{-3}), \\ \bar{h}_{ij} &= \mathcal{P}^{-1} \left(\left(-\frac{1}{\lambda} e^{\frac{2f}{\lambda}} \delta_{ij} + O(r^{-3}) \right) - \frac{1}{\lambda} \left(e^{\frac{2f}{\lambda}} \delta_{ij} + O(r^{-4}) \right) \right) = O(r^{-3}).\end{aligned}$$

Using the formula on the difference of the scalar curvatures of two metrics (c.f. [3], p187),

$$\begin{aligned}\mathcal{R}_{\bar{g}} &= \mathcal{R}_{\check{g}} + \sum_{s=1}^{\infty} (-1)^s \langle Ric_{\check{g}}, a^s \rangle_{\check{g}} + \frac{1}{4} \bar{g}^{ik} \bar{g}^{jl} \bar{g}^{pq} \left(-2 \check{\nabla}_i a_{lp} \check{\nabla}_j a_{kq} - 4 \check{\nabla}_i a_{kp} \check{\nabla}_j a_{lq} \right. \\ &\quad \left. + 4 \check{\nabla}_i a_{kl} \check{\nabla}_j a_{pq} + 3 \check{\nabla}_i a_{lq} \check{\nabla}_k a_{jp} - \check{\nabla}_i a_{jl} \check{\nabla}_k a_{pq} \right) - \bar{g}^{ik} \bar{g}^{jl} \left(\check{\nabla}_{i,k}^2 a_{jl} - \check{\nabla}_{i,l}^2 a_{jk} \right),\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{R}_{\bar{g}} &= -\bar{g}^{ik} \bar{g}^{jl} \left(\check{\nabla}_{i,k}^2 a_{jl} - \check{\nabla}_{i,l}^2 a_{jk} \right) + O(r^{-4}) \\ &= -\Delta_{\check{g}} a_{jj} + \check{\nabla}_{i,j}^2 a_{ij} + O(r^{-4}) \\ &= -2\Delta_{\check{g}} e^{\frac{2(f-t_0)}{\lambda}} + O(r^{-4}) \\ &= -\frac{4}{\lambda} e^{\frac{2(f-t_0)}{\lambda}} \Delta_{\check{g}} f + O(r^{-4}) = O(r^{-4}).\end{aligned}$$

This implies that \mathcal{R}_g is L^1 . Thus, (\mathbb{R}^3, g, K) is \mathcal{P} -asymptotically de Sitter with the constant $\mathcal{P} = e^{\frac{t_0}{\lambda}}$. Note that

$$\begin{aligned} \text{tr}_g(K) &= \frac{4}{\lambda}\mu - \frac{1}{\lambda}\mu^3 + \mu e^{-\frac{2f}{\lambda}} \left(\mu^2 e^{-\frac{2f}{\lambda}} \varepsilon^2 (1+r^2)^{-3} x_i x_j + \delta_{ij} \right) \\ &\quad \cdot \left(-\varepsilon (1+r^2)^{-\frac{3}{2}} \delta_{ij} + 3\varepsilon (1+r^2)^{-\frac{5}{2}} x_i x_j \right) \\ &= \frac{3}{\lambda} + \frac{1}{2\lambda} \varepsilon^2 r^2 (1+r^2)^{-3} e^{-\frac{2f}{\lambda}} + O(r^{-5}). \end{aligned}$$

This shows that if $\varepsilon \neq 0$, then $\text{tr}_g(K) > \frac{3}{\lambda}$ for large r , which violates the condition (2.1). Now the total energy is

$$\begin{aligned} E &= \frac{\mathcal{P}}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} (\partial_j \bar{g}_{ij} - \partial_i \bar{g}_{jj}) * dx^i \\ &= \frac{\mathcal{P}}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \left(\frac{4\varepsilon}{\lambda} r (1+r^2)^{-\frac{3}{2}} e^{\frac{2(f-t_0)}{\lambda}} - 3\mathcal{P}^{-2} \varepsilon^2 r (1+r^2)^{-3} \right) d\sigma_r \\ &= \frac{\varepsilon}{\lambda} e^{\frac{t_0}{\lambda}}. \end{aligned}$$

So $E < 0$ if $\varepsilon < 0$. Furthermore, the total linear momentum

$$P_k = \frac{\mathcal{P}^2}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} (\bar{h}_{ki} - \bar{g}_{ki} \text{tr}_{\bar{g}}(\bar{h})) * dx^i = \frac{\mathcal{P}^2}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} (O(r^{-3})) * dx^i = 0.$$

(ii) *The \mathcal{H} -asymptotically de Sitter case:* Consider the graph where

$$f(x) = T_0 + \varepsilon e^{\frac{-3R}{\lambda}}$$

for certain constant $T_0 \neq 0$ in the half-de Sitter spacetime equipped with the metric (1.3) in the hyperbolic coordinates. (Note that the hyperbolic coordinates are only valid for $T \neq 0$, $T_0 = 0$ can not be chosen here.) As

$$\rho^{-2} |\check{\nabla}^{\mathcal{H}} f|_{\check{g}_{\mathcal{H}}}^2 = \frac{9\varepsilon^2}{\lambda^2} e^{\frac{-6R}{\lambda}} \sinh^{-2} \frac{f}{\lambda},$$

(3.1) holds and the graph is an embedding spacelike hypersurface for ε is sufficiently small.

Recall that the standard hyperbolic metric $\check{g}_{\mathcal{H}}$ with the $\check{g}_{\mathcal{H}}$ -orthonormal frame $\check{e}_1, \check{e}_2, \check{e}_3$. By Cartan's structure equations $d\check{e}^i = -\omega_j^i \wedge \check{e}^j$ and $\Gamma_{ki}^j = \omega_j^i(\check{e}_k)$, we obtain

$$\begin{aligned} \omega_1^2 &= -\omega_2^1 = \cosh \frac{R}{\lambda} d\theta, & \omega_1^3 &= -\omega_3^1 = \cosh \frac{R}{\lambda} \sin \theta d\psi, & \omega_2^3 &= -\omega_3^2 = \cos \theta d\psi, \\ \Gamma_{21}^2 &= -\Gamma_{22}^1 = \frac{1}{\lambda} \coth \frac{R}{\lambda}, & \Gamma_{31}^3 &= -\Gamma_{33}^1 = \frac{1}{\lambda} \coth \frac{R}{\lambda}, & \Gamma_{32}^3 &= -\Gamma_{33}^2 = \frac{\cot \theta}{\lambda \sinh \frac{R}{\lambda}}, \end{aligned}$$

and other $\omega_j^i, \Gamma_{ij}^k$ vanish. Clearly, the gradient of f is

$$\check{\nabla}^{\mathcal{H}} f = \left(-\frac{3\varepsilon}{\lambda} e^{\frac{-3R}{\lambda}}, 0, 0, 0 \right).$$

Since

$$\begin{aligned}
\check{\nabla}_{1,1}^{\mathcal{H}} f &= \check{e}_1 \check{e}_1 f - \check{e}_i(f) \Gamma_{11}^i = \partial_R \partial_R f = \frac{9\varepsilon}{\lambda^2} e^{\frac{-3R}{\lambda}}, \\
\check{\nabla}_{2,2}^{\mathcal{H}} f &= \check{e}_2 \check{e}_2 f - \check{e}_i(f) \Gamma_{22}^i = -\check{e}_1(f) \Gamma_{22}^1 = -\frac{3\varepsilon}{\lambda^2} e^{\frac{-3R}{\lambda}} \coth \frac{R}{\lambda}, \\
\check{\nabla}_{3,3}^{\mathcal{H}} f &= \check{e}_3 \check{e}_3 f - \check{e}_i(f) \Gamma_{33}^i = -\check{e}_1(f) \Gamma_{33}^1 = -\frac{3\varepsilon}{\lambda^2} e^{\frac{-3R}{\lambda}} \coth \frac{R}{\lambda}, \\
\check{\nabla}_{1,2}^{\mathcal{H}} f &= \check{e}_1 \check{e}_2 f - \check{e}_i(f) \Gamma_{12}^i = 0, \\
\check{\nabla}_{1,3}^{\mathcal{H}} f &= \check{e}_1 \check{e}_3 f - \check{e}_i(f) \Gamma_{13}^i = 0, \\
\check{\nabla}_{2,3}^{\mathcal{H}} f &= \check{e}_2 \check{e}_3 f - \check{e}_i(f) \Gamma_{23}^i = 0,
\end{aligned}$$

we can find the Hessian of f

$$(\check{\nabla}_{i,j}^{\mathcal{H}} f) = \frac{3\varepsilon}{\lambda^2} e^{\frac{-3R}{\lambda}} \text{diag} \left(3, -\coth \frac{R}{\lambda}, -\coth \frac{R}{\lambda} \right).$$

Then the induced metric and the second fundamental forms of the graph are

$$\begin{aligned}
g &= \left(\sinh^2 \frac{f}{\lambda} - \frac{9\varepsilon^2}{\lambda^2} e^{\frac{-6R}{\lambda}} \right) dR^2 + \lambda^2 \sinh^2 \frac{f}{\lambda} \sinh^2 \frac{R}{\lambda} (d\theta^2 + \sin^2 \theta \psi^2), \\
K_{11} &= \mu \left(\frac{1}{\lambda} \cosh \frac{f}{\lambda} \sinh \frac{f}{\lambda} + \frac{9\varepsilon}{\lambda^2} e^{\frac{-3R}{\lambda}} - \frac{18\varepsilon^2}{\lambda^3} \coth \frac{f}{\lambda} e^{\frac{-6R}{\lambda}} \right), \\
&= \frac{1}{\lambda} \sinh \frac{T_0}{\lambda} \cosh \frac{T_0}{\lambda} + \frac{\varepsilon}{\lambda^2} \left(9 + \sinh^2 \frac{T_0}{\lambda} + \cosh^2 \frac{T_0}{\lambda} \right) e^{\frac{-3R}{\lambda}} + O \left(e^{\frac{-6R}{\lambda}} \right), \\
K_{22} &= K_{33} = \mu \left(\frac{1}{\lambda} \cosh \frac{f}{\lambda} \sinh \frac{f}{\lambda} - \frac{3\varepsilon}{\lambda^2} e^{\frac{-3R}{\lambda}} \coth \frac{R}{\lambda} \right) \\
&= \frac{1}{\lambda} \sinh \frac{T_0}{\lambda} \cosh \frac{T_0}{\lambda} + \frac{\varepsilon}{\lambda^2} \left(-3 + \sinh^2 \frac{T_0}{\lambda} + \cosh^2 \frac{T_0}{\lambda} \right) e^{\frac{-3R}{\lambda}} + O \left(e^{\frac{-5R}{\lambda}} \right),
\end{aligned}$$

and other $K_{ij} = 0$, where $\mu = \left(1 - \frac{9\varepsilon^2}{\lambda^2} e^{\frac{-6R}{\lambda}} \sinh^{-2} \frac{f}{\lambda} \right)^{-\frac{1}{2}} = 1 + O \left(e^{\frac{-6R}{\lambda}} \right)$. Set $\mathcal{H} = \sinh \frac{T_0}{\lambda}$, $\bar{g} = \mathcal{H}^{-2} g$, $a = \bar{g} - \check{g}_{\mathcal{H}}$ and $\bar{h} = \mathcal{H}^{-1} \left(K - \frac{\coth \frac{T_0}{\lambda}}{\lambda} g \right)$, we obtain

$$\begin{aligned}
(a_{ij}) &= \frac{2\varepsilon}{\lambda} \coth \frac{T_0}{\lambda} e^{\frac{-3R}{\lambda}} \text{diag} \left(1 + O(e^{\frac{-3R}{\lambda}}), 1 + O(e^{\frac{-3R}{\lambda}}), 1 + O(e^{\frac{-3R}{\lambda}}) \right), \\
(\bar{h}_{ij}) &= \frac{4\varepsilon}{\lambda^2} \sinh^{-1} \frac{T_0}{\lambda} e^{\frac{-3R}{\lambda}} \text{diag} \left(2 + O(e^{\frac{-3R}{\lambda}}), -1 + O(e^{\frac{-2R}{\lambda}}), -1 + O(e^{\frac{-2R}{\lambda}}) \right), \\
\check{\nabla}_k^{\mathcal{H}} a_{ij} &= O \left(e^{\frac{-3R}{\lambda}} \right), \quad \check{\nabla}_{k,l}^{\mathcal{H}} a_{ij} = O \left(e^{\frac{-3R}{\lambda}} \right), \quad \check{\nabla}_k^{\mathcal{H}} \bar{h}_{ij} = O \left(e^{\frac{-3R}{\lambda}} \right).
\end{aligned}$$

Now we compute the scalar curvature. Note that

$$\begin{aligned}
tr_{\check{g}_{\mathcal{H}}} a &= a_{11} + a_{22} + a_{33} = \frac{6\varepsilon}{\lambda} \coth \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} + O(e^{-\frac{6R}{\lambda}}), \\
\check{\nabla}_{i,i}^{\mathcal{H}} a_{jj} &= \frac{6\varepsilon}{\lambda} \coth \frac{T_0}{\lambda} \Delta_{\check{g}_{\mathcal{H}}} e^{-\frac{3R}{\lambda}} + O(e^{-\frac{6R}{\lambda}}), \\
\check{\nabla}_{i,j}^{\mathcal{H}} a_{ij} &= \check{\nabla}_{i,j}^{\mathcal{H}} \left(\frac{2\varepsilon}{\lambda} \coth \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} \check{g}_{\mathcal{H}ij} + O(e^{-\frac{6R}{\lambda}}) \right) = \frac{2\varepsilon}{\lambda} \coth \frac{T_0}{\lambda} \Delta_{\check{g}_{\mathcal{H}}} e^{-\frac{3R}{\lambda}} + O(e^{-\frac{6R}{\lambda}}), \\
\Delta_{\check{g}_{\mathcal{H}}} e^{-\frac{3R}{\lambda}} &= \frac{3}{\lambda^2} e^{-\frac{3R}{\lambda}} + O(e^{-\frac{5R}{\lambda}}),
\end{aligned}$$

We have

$$\begin{aligned}
\mathcal{R}_{\bar{g}} &= \mathcal{R}_{\check{g}_{\mathcal{H}}} - \langle Ric_{\check{g}_{\mathcal{H}}}, a \rangle_{\check{g}_{\mathcal{H}}} - \check{\nabla}_{i,i}^{\mathcal{H}} a_{jj} + \check{\nabla}_{i,j}^{\mathcal{H}} a_{ij} + O(e^{-\frac{6R}{\lambda}}) \\
&= -\frac{6}{\lambda^2} + \frac{12\varepsilon}{\lambda^3} \coth \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} - \frac{18\varepsilon}{\lambda^3} \coth \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} + \frac{6\varepsilon}{\lambda^3} \coth \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} + O(e^{-\frac{5R}{\lambda}}) \\
&= -\frac{6}{\lambda^2} + O(e^{-\frac{5R}{\lambda}}).
\end{aligned}$$

So $(\mathcal{R}_{\bar{g}} + \frac{6}{\lambda^2}) e^{\frac{\rho_{\bar{g}}}{\lambda}}$ is L^1 with respect to the metric \bar{g} . Next, we shall show that $(\bar{\nabla}^j \bar{h}_{ij} - \bar{\nabla}_i tr_{\bar{g}}(\bar{h})) e^{\frac{R}{\lambda}}$ is $L^1(\mathbb{R}^3, \check{g}_{\mathcal{H}})$. As

$$\begin{aligned}
\check{\nabla}_i^{\mathcal{H}} tr_{\check{g}_{\mathcal{H}}}(\bar{h}) &= O(e^{-\frac{5R}{\lambda}}), \\
\sum_j \check{\nabla}_j^{\mathcal{H}} \bar{h}_{1j} &= \check{\partial}_j(\bar{h}_{1j}) - \bar{h}_{kj} \Gamma_{j1}^k - \bar{h}_{1k} \Gamma_{jj}^k \\
&= \check{\partial}_1(\bar{h}_{11}) - \bar{h}_{22} \Gamma_{21}^2 - \bar{h}_{33} \Gamma_{31}^3 - \bar{h}_{11}(\Gamma_{22}^1 + \Gamma_{33}^1) \\
&= \partial_R \left(\frac{8\varepsilon}{\lambda^2} \sinh^{-1} \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} \right) + \frac{4\varepsilon}{\lambda^2} \sinh^{-1} \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} \left(\frac{2}{\lambda} \coth \frac{R}{\lambda} \right) \\
&\quad - \frac{8\varepsilon}{\lambda^2} \sinh^{-1} \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} \left(-\frac{2}{\lambda} \coth \frac{R}{\lambda} \right) + O(e^{-\frac{6R}{\lambda}}) = O(e^{-\frac{5R}{\lambda}}), \\
\sum_j \check{\nabla}_j^{\mathcal{H}} \bar{h}_{2j} &= \check{\partial}_j(\bar{h}_{2j}) - \bar{h}_{kj} \Gamma_{j2}^k - \bar{h}_{2k} \Gamma_{jj}^k \\
&= -\bar{h}_{jj} \Gamma_{j2}^j - \bar{h}_{22} \Gamma_{jj}^2 + O(e^{-\frac{5R}{\lambda}}) = O(e^{-\frac{5R}{\lambda}}), \\
\sum_j \check{\nabla}_j^{\mathcal{H}} \bar{h}_{3j} &= \check{\partial}_j(\bar{h}_{3j}) - \bar{h}_{kj} \Gamma_{j3}^k - \bar{h}_{3k} \Gamma_{jj}^k \\
&= -\bar{h}_{jj} \Gamma_{j3}^j - \bar{h}_{33} \Gamma_{jj}^3 + O(e^{-\frac{5R}{\lambda}}) = O(e^{-\frac{5R}{\lambda}}),
\end{aligned}$$

we obtain that $(\check{\nabla}^j \bar{h}_{ij} - \check{\nabla}_i tr_{\check{g}_{\mathcal{H}}}(\bar{h})) e^{\frac{R}{\lambda}}$ is $L^1(\mathbb{R}^3, \check{g}_{\mathcal{H}})$. Denote symmetric 2-tensor

$$\Gamma(X, Y) = \bar{\nabla}_X Y - \check{\nabla}_X^{\mathcal{H}} Y. \text{ There is constant } C_1 \text{ such that } |\Gamma|_{\check{g}_{\mathcal{H}}} \leq C_1 \frac{|\check{\nabla}^{\mathcal{H}} a|_{\check{g}_{\mathcal{H}}}}{1 - |a|_{\check{g}_{\mathcal{H}}}} = O(e^{-\frac{3R}{\lambda}})$$

(c.f. [3], p188). Therefore,

$$|\bar{\nabla} \bar{h} - \check{\nabla}^{\mathcal{H}} \bar{h}|_{\check{g}_{\mathcal{H}}} \leq C_2 |\bar{h}|_{\check{g}_{\mathcal{H}}} |\Gamma|_{\check{g}_{\mathcal{H}}} = O\left(e^{-\frac{6R}{\lambda}}\right),$$

and we conclude that $(\bar{\nabla}^j \bar{h}_{ij} - \bar{\nabla}_i \text{tr}_{\bar{g}}(\bar{h}))e^{\frac{R}{\lambda}}$ is $L^1(\mathbb{R}^3, \check{g}_{\mathcal{H}})$. Thus (\mathbb{R}^3, g, K) is \mathcal{H} -asymptotically de Sitter with the constant $\mathcal{H} = \sinh \frac{T_0}{\lambda}$. Now

$$\begin{aligned} \text{tr}_g(K) &= \mu \sinh^{-2} \frac{f}{\lambda} \left(\delta_{ij} + \mu^2 \sinh^{-2} \frac{f}{\lambda} \check{\nabla}_i^{\mathcal{H}} f \check{\nabla}_j^{\mathcal{H}} f \right) \check{\nabla}_{i,j}^{\mathcal{H}} f \\ &\quad - \frac{1}{\lambda} \mu^3 \coth \frac{f}{\lambda} + \frac{4}{\lambda} \mu \coth \frac{f}{\lambda} \\ &= \frac{3}{\lambda} \coth \frac{T_0}{\lambda} - \frac{12\varepsilon}{\lambda^2} \sinh^{-2} \frac{T_0}{\lambda} e^{-\frac{5R}{\lambda}} + O\left(e^{-\frac{6R}{\lambda}}\right). \end{aligned}$$

Therefore,

$$\text{tr}_g(K) \sinh \frac{T_0}{\lambda} - \frac{3}{\lambda} \cosh \frac{T_0}{\lambda} = -\frac{12\varepsilon}{\lambda^2 \sinh \frac{T_0}{\lambda}} e^{-\frac{5R}{\lambda}} + O\left(e^{-\frac{6R}{\lambda}}\right).$$

Now we calculate the total energy-momentum. As

$$\begin{aligned} \mathcal{E} &= \check{\nabla}^{\mathcal{H},j} \bar{g}_{1j} - \check{\nabla}_1^{\mathcal{H}} \text{tr}_{\check{g}_{\mathcal{H}}}(\bar{g}) + \frac{1}{\lambda} (a_{22} + a_{33}) + 2(\bar{h}_{22} + \bar{h}_{33}) \\ &= -\frac{6\varepsilon}{\lambda^2} \coth \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} + \frac{18\varepsilon}{\lambda^2} \coth \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} \\ &\quad + \frac{4\varepsilon}{\lambda^2} \coth \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} - \frac{16\varepsilon}{\lambda^2} \sinh^{-1} \frac{T_0}{\lambda} e^{-\frac{3R}{\lambda}} + O\left(e^{-\frac{5R}{\lambda}}\right) \\ &= \frac{16\varepsilon}{\lambda^2} \tanh \frac{T_0}{2\lambda} e^{-\frac{3R}{\lambda}} + O\left(e^{-\frac{5R}{\lambda}}\right), \end{aligned}$$

we have

$$\begin{aligned} E_{\nu}^{\mathcal{H}} &= \frac{\mathcal{H}^2}{16\pi} \lim_{R \rightarrow \infty} \int_{S_R} \mathcal{E} n^{\nu} e^{\frac{R}{\lambda}} \check{\varepsilon}^2 \wedge \check{\varepsilon}^3 \\ &= \frac{\sinh^2 \frac{T_0}{\lambda}}{16\pi} \lim_{R \rightarrow \infty} \int_{S_R} \frac{16\varepsilon}{\lambda^2} \tanh \frac{T_0}{2\lambda} e^{-\frac{3R}{\lambda}} n^{\nu} e^{\frac{R}{\lambda}} \lambda^2 \sinh^2 \frac{R}{\lambda} \sin \theta d\theta \wedge d\psi \\ &= \frac{\varepsilon}{4\pi} \tanh \frac{T_0}{2\lambda} \sinh^2 \frac{T_0}{\lambda} \int_{S_1} n^{\nu} \sin \theta d\theta \wedge d\psi. \end{aligned}$$

Since $(n^{\nu}) = (1, \sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta)$, we obtain

$$E_{\nu}^{\mathcal{H}} = \begin{cases} \varepsilon \tanh \frac{T_0}{2\lambda} \sinh^2 \frac{T_0}{\lambda}, & \text{if } \nu = 0, \\ 0, & \text{if } \nu = 1, 2, 3. \end{cases}$$

So $E_0^{\mathcal{H}} < 0$ if $\varepsilon T_0 < 0$ and in this case (2.2) does not hold for large R .

We remark that the de Sitter metric (1.3) collapses at $T = 0$ which can be viewed as the big bang of the universe. It is interesting to find, at the big bang, that

$$\lim_{T_0 \rightarrow 0} E_0^{\mathcal{H}} = 0.$$

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